

A multipartite version of the Hajnal-Szemerédi theorem for graphs and hypergraphs *

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Abstract

A perfect K_t -matching in a graph G is a spanning subgraph consisting of vertex disjoint copies of K_t . A classic theorem of Hajnal and Szemerédi [8] states that if G is a graph of order n with minimum degree $\delta(G) \geq (t-1)n/t$ and $t|n$, then G contains a perfect K_t -matching. Let G be a t -partite graph with vertex classes V_1, \dots, V_t each of size n . We show that if every vertex $x \in V_i$ is joined to at least $((t-1)/t + \gamma)n$ vertices of V_j for $i \neq j$, then G contains a perfect K_t -matching, thus verifying a conjecture of Fisher asymptotically [6]. Furthermore, we consider a generalisation to hypergraphs in terms of the codegree.

1 Introduction

Given a graph G and an integer $t \geq 3$, a K_t -matching is a set of vertex disjoint copies of K_t in G . A *perfect K_t -matching* (or K_t -factor) is a spanning K_t -matching. Clearly, if G contains a perfect K_t -matching then t divides $|G|$. A classic theorem of Hajnal and Szemerédi [8] states a relationship between the minimum degree and the existence of a perfect K_t -matching.

Theorem 1.1 (Hajnal-Szemerédi Theorem [8]). *Let $t > 2$ be an integer. Let G be a graph of order n with minimum degree $\delta(G) \geq (t-1)n/t$ and $t|n$. Then G contains a perfect K_t -matching.*

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Let G be a t -partite graph with vertex classes V_1, \dots, V_t . We say that G is *balanced* if $|V_i| = |V_j|$ for $1 \leq i < j \leq t$. Write $G[V_i, V_j]$ to be the induced bipartite subgraph on vertex classes V_i and V_j . Define $\tilde{\delta}(G)$ to be $\min_{1 \leq i < j \leq t} \delta(G[V_i, V_j])$. Fischer [6] conjectured the following multipartite version of the Hajnal-Szemerédi theorem.

Conjecture 1.2 (Fischer [6]). *Let G be a balanced t -partite graph with each class of size n . Then there exists a positive constant K such that if $\tilde{\delta}(G) \geq (t-1)n/t + K$, then G contains a perfect K_t -matching.*

Note that the constant K was not stated in Fischer's original conjecture, but it was shown to be necessary for odd t in [17]. For $k = 2$, the conjecture can be easily verified by Hall's Theorem. For $k = 3$, Johansson [10] proved that $\tilde{\delta}(G) \geq 2n/3 + \sqrt{n}$ suffices for all n . Using the regularity lemma, Magyar and Martin [17], and Martin and Szemerédi [18] proved the exact result for $k = 3$ and $k = 4$ respectively for n sufficiently large. For $k \geq 5$, Csaba and Mydlarz [4] proved that $\tilde{\delta}(G) \geq c_t n / (c_t + 1)$ is sufficient, where $c_t = t - 3/2 + (1 + 1/2 + \dots + 1/t)/2$. In this paper, we show that Conjecture 1.2 is true asymptotically.

Theorem 1.3. *Let $t \geq 2$ be an integer and let $\gamma > 0$. Then there exists an integer $n_0 = n_0(t, \gamma)$ such that if G is a balanced t -partite graph with each class of size $n \geq n_0$ and $\tilde{\delta}(G) \geq ((t-1)/t + \gamma)n$, then G contains a perfect K_t -matching.*

Independently, Theorem 1.3 also has been proved by Keevash and Mycroft [12]. Their proof involves the hypergraph blowup lemma [11], so n_0 is extremely large whereas our proof gives a much smaller n_0 .

We further generalise Theorem 1.3 to hypergraphs. For $a \in \mathbb{N}$, we refer to the set $\{1, \dots, a\}$ as $[a]$. For a set U , we denote by $\binom{U}{k}$ the set of k -sets of U . A k -uniform hypergraph, k -graph for short, is a pair $H = (V(H), E(H))$, where $V(H)$ is a finite set of vertices and $E(H) \subset \binom{V(H)}{k}$ is a family of k -sets of $V(H)$. We simply write V to mean $V(H)$ if it is clear from the context. For a k -graph H and an l -set $T \in \binom{V}{l}$, let $N^H(T)$ be the set of $(k-l)$ -sets $S \in \binom{V}{k-l}$ such that $S \cup T$ is an edge in H . Let $\deg^H(T) = |N^H(T)|$. Define the *minimum l -degree* $\delta_l(H)$ of H to be the minimal $\deg^H(T)$ over all $T \in \binom{V}{l}$. For $U \subset V$, we denote by $H[U]$ the induced subgraph of H on vertex set U .

A k -graph H is *t -partite*, if there exists a partition of the vertex set V into t classes V_1, \dots, V_t such that every edge intersects every class in at most one vertex. Similarly, H is *balanced* if $|V_1| = \dots = |V_t|$. An l -set $T \in \binom{V}{l}$ is said to be *legal* if $|T \cap V_i| \leq 1$ for $i \in [t]$. For $I \subset [t]$, $T \subset V$ is *I -legal* if $|T \cap V_i| = 1$ for $i \in I$ and $|T \cap V_i| = 0$ otherwise. We write V_I to be the set of I -legal sets. For $I \cup J \in \binom{[t]}{k}$ with $I \cap J = \emptyset$ and an I -legal set $T \in V_I$, denote by $N_J^H(T)$ be the set of J -legal sets $S \in V_J$ such that $S \cup T$ is an edge in H and write $\deg_J^H(T) = |N_J^H(T)|$. For $l \in [k-1]$ and $I \in \binom{[t]}{l}$, define $\tilde{\delta}_I(H) = \min\{\deg_J^H(T) : T \in V_I \text{ and } J \in \binom{[t] \setminus I}{k-l}\}$. Finally, we set $\tilde{\delta}_l(H) = \min\{\tilde{\delta}_I(H) : I \in \binom{[t]}{l}\}$. If H is clear from the context, we drop the suffix of H .

Let K_t^k be the complete k -graph on t vertices. It is easy to see that a t -partite k -graph H contains a perfect K_t^k -matching only if H is balanced.

Definition 1.4. *Let $1 \leq l < k \leq t$ and $n \geq 1$ be integers. Define $\phi_l^k(t, n)$ to be the smallest integer d such that every t -partite k -graph H with each class of size n and $\tilde{\delta}_l(H) \geq d$ contains a perfect K_t^k -matching.*

Equivalently,

$$\phi_l^k(t, n) = \min\{d : \forall \widetilde{\delta}_l(H) \geq d \Rightarrow H \text{ contains a perfect } K_t^k\text{-matching}\},$$

where H is a t -partite k -graph H with each class of size n . Write $\phi^k(t, n)$ for $\phi_{k-1}^k(t, n)$.

Note that Theorem 1.3 implies that $\phi^2(t, n) \sim (t-1)n/t$. Various cases of $\phi_l^k(k, n)$ have been studied. Daykin and Häggkvist [5] showed that $\phi_1^k(k, n) \leq (k-1)n^{k-1}/k$, which was later improved by Hán, Person and Schacht [9]. Kühn and Osthus [13] showed that $n/2 - 1 < \phi^k(k, n) = \phi_{k-1}^k(k, n) \leq n/2\sqrt{2n \log n}$. Aharoni, Georgakopoulos and Sprüssel [1] then reduced the upper bound to $\phi^k(k, n) \leq \lceil (n+1)/2 \rceil$. For $k/2 \leq l < k-1$, Pikhurko [19] showed that $\phi_l^k(k, n) \leq n^{k-l}/2$. The exact value of $\phi_1^3(3, n)$ has been determined by the authors in [15]. In this paper, we give an upper bound on $\phi^k(t, n)$ for $3 \leq k < t$.

Theorem 1.5. *For $3 \leq k < t$ and $\gamma \geq 0$, there exists an integer $n_0 = n_0(k, t, \gamma)$ such that for all $n \geq n_0$*

$$\phi^k(t, n) \leq \left(1 - \left(\binom{t-1}{k-1} + 2\binom{t-2}{k-2}\right)^{-1} + \gamma\right)n.$$

Our proofs of Theorem 1.3 and Theorem 1.5 use the absorption technique introduced by Rödl, Ruciński and Szemerédi [20]. We now present an outline of the absorption technique. First, we remove a set U of disjoint copies of K_t^k from H satisfying the conditions of the absorption lemma, Lemma 3.2, and call the resulting graph H' . Next, we find a K_t^k -matching covering almost all vertices of H' . Let W be the set of ‘leftover’ vertices. By the absorption property of U , there is a perfect K_t^k -matching in $H[U \cup W]$. Hence, we obtain a perfect K_t^k -matching in H as required.

In order to find a K_t^k -matching covering almost all vertices of H' , we follow the approach of Alon et al. [2], which consider fractional matchings. Let $\mathcal{K}_t^k(H)$ be the set of K_t^k in a k -graph H . A *fractional K_t^k -matching* in a k -graph H is a function $w : \mathcal{K}_t^k(H) \rightarrow [0, 1]$ such that for each $v \in V$ we have

$$\sum \{w(T) : v \in T \in \mathcal{K}_t^k(H)\} \leq 1.$$

Then $\sum_{T \in \mathcal{K}_t^k(H)} w(T)$ is the *size* of w . If the size is $|H|/t$, then w is *perfect*. We are interested in perfect fractional K_t^k -matchings w in t -partite k -graph H with each class of size n . Note that $|H| = tn$, so if w is a perfect fractional K_t^k -matching in H , then

$$\sum \{w(T) : v \in T \in \mathcal{K}_t^k(H)\} \leq 1 \text{ for } v \in V \text{ and } \sum_{T \in \mathcal{K}_t^k(H)} w(T) = n.$$

Define $\phi_l^{*,k}(t, n)$ to be the fractional analogue of $\phi_l^k(t, n)$.

Theorem 1.6. *For $2 \leq k \leq t$ and $n \geq 1$,*

$$\lceil (t-k+1)n/t \rceil \leq \phi^{*,k}(t, n) \leq \begin{cases} \lceil (t-1)n/t \rceil & \text{for } k=2, \\ \left\lceil \left(1 - \binom{t-1}{k-1}^{-1}\right)n \right\rceil + 1 & \text{for } k \geq 3. \end{cases}$$

In particular, $\phi^{,2}(t, n) = \lceil (t-1)n/t \rceil$.*

Notice that Theorem 1.6 is only tight for $k = 2$. The upper bound on $\phi^{*,k}(t, n)$ given in Theorem 1.6 is sufficient for our purpose, that is, to prove Theorem 1.3 and Theorem 1.5. In addition, we also obtain the following result.

Theorem 1.7. *Let $2 \leq k \leq t$ be integers with $\epsilon > 0$. Then, there exist an integer n_0 and a constant $\gamma > 0$ such that every k -graph H of order $n > n_0$ and*

$$\delta_{k-1}(H) \geq t\phi^{*,k}(t, \lceil n/t \rceil) + \gamma n$$

contains a K_t^k -matching \mathcal{T} in H covering all but at most ϵn vertices.

Together with Theorem 1.6, we obtain the following corollary for general k -graphs.

Corollary 1.8. *Let $3 \leq k \leq t$ be integers with $\epsilon > 0$. Then, there exist an integer n_0 and a constant $\gamma > 0$ such that every k -graph H of order $n > n_0$ and*

$$\delta_{k-1}(H) \geq \left(1 - \binom{t-1}{k-1}^{-1} + \gamma\right) n$$

contains a K_t^k -matching \mathcal{T} in H covering all but at most ϵn vertices.

Observe that Corollary 1.8 is a stronger statement than Lemma 6.1 in [14]. Thus, by replacing Lemma 6.1 in [14] with Theorem 1.7, we improve the bounds of Theorem 1.4 in [14].

In the next section, we prove Theorem 1.6. Both Theorem 1.3 and Theorem 1.5 are proved simultaneously in Section 3. Finally, Theorem 1.7 is proved in Section 4.

2 Perfect fractional K_t^k -matchings

In this section we are going to prove Theorem 1.6. We require Farkas Lemma.

Lemma 2.1 (Farkas Lemma (see [16] P.257)). *A system of equations $yA = b$, $y \geq 0$ is solvable if and only if the system $Ax \geq 0$, $bx < 0$ is unsolvable.*

First we prove the lower bounds on $\phi^{*,k}(t, n)$.

Proposition 2.2. *Let $2 \leq k \leq t$ and $n \geq 1$ be integers. There exists a t -partite k -graph H with each class of size n with $\tilde{\delta}_{k-1}(H) = \lceil (t - k + 1)n/t \rceil - 1$ without a perfect fractional K_t^k -matching.*

Proof. We fix t, k and n . Let V_1, \dots, V_t be disjoint vertex set each of size n . For $i \in [t]$, fixed a $(\lceil (t - k + 1)n/t \rceil - 1)$ -set $W_i \subset V_i$. Define H be the t -partite k -graph on vertex classes V_1, \dots, V_t such that every edge in H meets W_i for some i . Clearly, $\tilde{\delta}_{k-1}(H) = \lceil (t - k + 1)n/t \rceil - 1$. Thus, it suffices to show that H does not contain a perfect fractional K_t^k -matching. Let A be the matrix of H with rows representing the $K_t^k(H)$ and the columns representing the vertices of H such that $A_{T,v} = 1$ for $T \in \mathcal{K}_t^k(H)$ and $v \in V$ if and only if $v \in T$. By Farkas Lemma, Lemma 2.1 taking $y = (w(T) : T \in \mathcal{K}_t^k(H))$ and $b = (1, \dots, 1)$, there is no perfect

fractional K_t^k -matching in H if and only if there is a weighting function $w : V \rightarrow \mathbb{R}$ such that

$$\forall T \in \mathcal{K}_t^k(H) \sum_{v \in T} w(v) \geq 0 \text{ and } \sum_{v \in V} w(v) < 0. \quad (1)$$

Set $w(v) = (k-1)/(t-k+1)$ if $v \in \bigcup_{i \in [q]} W_i$ and $w(v) = -1$ otherwise. Clearly,

$$\sum w(v) = \frac{k-1}{t-k+1} t \left(\left\lceil \frac{(t-k+1)n}{t} \right\rceil - 1 \right) - t \left(n - \left\lceil \frac{(t-k+1)n}{t} \right\rceil + 1 \right) < 0.$$

For $T \in \mathcal{K}_t^k(H)$, T contains at least $t-k+1$ vertices of in $\bigcup_{i \in [t]} W_i$ and so $\sum_{v \in T} w(v) \geq 0$. Thus, w satisfies (1), so H does not contain a perfect fractional K_t^k -matching. \square

Proof of Theorem 1.6. By Proposition 2.2, it is sufficient to prove the upper bound on $\phi^{*,k}(t, n)$. Fix k , t and n . Suppose the contrary that there exists a t -partite k -graph H with each class of size n and

$$\tilde{\delta}_{k-1}(H) \geq \tilde{\delta}$$

that does not contain a perfect fractional K_t^k -matching, where $\tilde{\delta}$ is the upper bound on $\phi^{*,k}(t, n)$ suggested by the theorem. By a similar argument, as in the proof of Proposition 2.2, there is a weighting function $w : V \rightarrow \mathbb{R}$ satisfying (1). Let V_1, \dots, V_n be the vertex classes of H with $V_i = \{v_{i,1}, \dots, v_{i,n}\}$ for $i \in [t]$. We identify the t -tuple $(j_1, \dots, j_t) \in [n]^t$ to be the $[t]$ -legal set $\{v_{1,j_1}, \dots, v_{t,j_t}\}$ and write $w(j_1, \dots, j_t)$ to mean $\sum_{i \in [t]} w(v_{i,j_i})$. Without loss of generality we may assume that $w(v_{i,j}) \geq w(v_{i,j'})$ for $1 \leq j < j' \leq n$ and $i \in [t]$. By considering the vertex weighting w' such that

$$w'(v) = \begin{cases} w(v) + \epsilon & \text{if } v \in V_i, \\ w(v) - \epsilon & \text{if } v \in V_{i'}, \\ w(v) & \text{otherwise,} \end{cases}$$

with $\epsilon > 0$, we may assume that $w(v_{i,n}) = w(v_{i',n})$ for $i, i' \in [t]$. Furthermore, we may assume that $w(v_{i,n}) = w(v_{i',n}) = -1$ for $i, i' \in [t]$ and so $w(v) \leq t-1$ for all $v \in V$. Finally, we apply the linear transformation $(w(v) + 1)/t$ for $v \in V$, which scales w so that it now lies in the interval $[0, 1]$ and w satisfies the following inequalities

$$\forall T \in \mathcal{K}_t^k(H) \sum_{v \in T} w(v) \geq 1 \text{ and } \sum_{v \in V} w(v) < n. \quad (2)$$

For $j \in [t]$, set $r(j) = n - \binom{j-1}{k-1}(n - \tilde{\delta})$ for $j \in [t]$. Note that $r(j) = n$ for $j \in [k-1]$ and $r(k) = \tilde{\delta}$. By the definition of $\tilde{\delta}$, we know that $r(t) \geq 1$. Thus, for a J -legal set $T \in \mathcal{K}_j^k(H)$ with $J \in \binom{[t]}{j}$ and $j < k$, there are at least $r(j+1)$ vertices $v \in V_i$ such that $T \cup v$ forms a K_{j+1}^k for each $i \in [t] \setminus J$. Hence, we can find a $K_t^k(j_1, j_2, \dots, j_q)$ with $j_i \geq r(i)$ for $i \in [t]$. Recall that $w(v_{i,j}) \geq w(v_{i,j'})$ for $i \in [t]$ and $1 \leq j < j' \leq n$. Therefore,

$$\sum_{i \in [t]} w(v_{i,r(i)}) = w(r(1), r(2), \dots, r(t)) \geq w(j_1, j_2, \dots, j_t) \geq 1$$

by (2). By a similar argument, we have

$$\sum_{i \in [t]} w(v_{i, r(\sigma(i))}) \geq 1,$$

where σ is a permutation on $[t]$. By setting $\sigma = (1, 2, \dots, t)$, we have

$$\sum_{i \in [t]} \sum_{j \in [t]} w(v_{i, r(j)}) = \sum_{j \in [t]} \sum_{i \in [t]} w(v_{i, r(\sigma^j(i))}) \geq t. \quad (3)$$

Observe that $w(v_{i, r(j)}) \leq w(v_{i, r(j+1)})$ for $i \in [t]$ and $j \in [t-1]$. Since $r(j) = n$ for $j \in [k-1]$ and $w(v_{i, n}) = 0$ for $i \in [t]$,

$$\begin{aligned} \sum_{i \in [t]} w(v_{i, r(t)}) &= \frac{1}{t-k+1} \sum_{i \in [t]} \left(\sum_{j \in [k-1]} w(v_{i, r(j)}) + (t-k+1)w(v_{i, r(t)}) \right) \\ &\geq \frac{1}{t-k+1} \sum_{i \in [t]} \sum_{j \in [t]} w(v_{i, r(j)}) \geq \frac{t}{t-k+1}, \end{aligned} \quad (4)$$

where the last inequality is due to (3).

Claim 2.3.

$$\sum_{i \in [t]} \left(\sum_{j \in [t-1]} (r(j) - r(j+1))w(v_{i, r(j)}) + \frac{r(k) - r(t)}{t-k} w(v_{i, r(t)}) \right) \geq \frac{t(r(k) - r(t))}{t-k}.$$

Proof of claim. Consider the multiset A containing $(t-k)(r(j) - r(j+1))$ copies of j for $k \leq j \leq t-1$ and $r(k) - r(j)$ copies of t . We claim that there exists a partition of A into $(t-k+1)$ -sets B such that

- (i) there is exactly one copy of t in B ,
- (ii) the number of j in B with $j \leq k+i-1$ is at most i for $i \in [t-k]$.

First note that

$$\sum_{k \leq j \leq t-1} (r(j) - r(j+1)) = r(k) - r(t).$$

Hence, (i) easily can be easily satisfied. Note that $r(j) - r(j+1) = \binom{j-2}{t-1}(n-\delta)$. Hence, there are more copies of j' than copies of j in A for $k \leq j < j' \leq t-1$. Now greedily pick $(t-k+1)$ -set $B = \{b_1, \dots, b_{k-t+1}\}$ satisfying (i) and (ii) such that $\sum b_i$ is minimal. It can be easily verified that we would obtain a partition of A as claimed. Fix one such partition \mathcal{B} . Recall that $w(v_{i, r(j)}) \leq w(v_{i, r(j+1)})$ for $i \in [t]$ and $j \in [t-1]$ and $w(v_{i, r(j)}) = w(v_{i, n}) = 0$ for $i \in [t]$ and $j \in [k-1]$. For $B = \{b_1, \dots, b_{k-t+1}\} \in \mathcal{B}$, we have

$$\sum_{i \in [t]} \sum_{j \in [k-t+1]} w(v_{i, r(b_j)}) \geq \sum_{i \in [t]} \sum_{k \leq j \leq t} w(v_{i, r(j)}) = \sum_{i \in [t]} \sum_{j \in [t]} w(v_{i, r(j)}) \geq t$$

by (3). Hence, Claim 2.3 follows easily by summing the above inequality over all $B \in \mathcal{B}$. \square

Recall that $r(k) = \tilde{\delta}$ and $r(1) = n$. By Claim 2.3 and (4), we have

$$\begin{aligned}
\sum_{i \in [t]} \sum_{j \in [n]} w(v_{i,j}) &\geq \sum_{i \in [t]} \left(\sum_{j \in [t-1]} (r(j) - r(j+1)) w(v_{i,r(j)}) + r(t) w(v_{i,r(t)}) \right) \\
&\geq \frac{t(r(k) - r(t))}{t-k} + \sum_{i \in [t]} \left(r(t) - \frac{r(k) - r(t)}{t-k} \right) w(v_{i,r(t)}) \\
&\geq \frac{t(r(k) - r(t))}{t-k} + \left(r(t) - \frac{r(k) - r(t)}{t-k} \right) \frac{t}{t-k+1} \\
&= \frac{tr(k)}{t-k+1} = \frac{t\tilde{\delta}}{t-k+1} \geq n
\end{aligned}$$

contradicting (2). The proof of Theorem 1.6 is completed. \square

3 Proof of Theorem 1.3 and Theorem 1.5

First we would need the following simple proposition.

Proposition 3.1. *Let $\gamma > 0$. Let H be a balanced t -partite k -graph with partition classes V_1, \dots, V_t , each of size n and*

$$\tilde{\delta}_{k-1}(H) \geq \left(1 - \left(\binom{t-2}{k-1} + 2 \binom{t-2}{k-2} \right)^{-1} + \gamma \right) n.$$

Then, for $i \in [t]$ and distinct vertices $u, v \in V_i$, there are at least $(\gamma n)^{t-1}$ legal $[t] \setminus i$ -sets T such that $T \cup u$ and $T \cup v$ are K_t^k in H .

Proof. Let $u, v \in V_1$. For $2 \leq i \leq t$, we pick $w_i \in V_i$ such that $w_i \in N(T)$ for all legal $(k-1)$ -sets $T \subset \{u, v, w_2, \dots, w_{i-1}\}$. By the definition of $\tilde{\delta}_{k-1}(H)$, there are at least γn choices for each w_i . The proposition easily follows. \square

Using Proposition 3.1, we obtain an absorption lemma. Its proof can be easily obtained by modifying the proof of Lemma 4.2 in [15]. For the sake of completeness, it is included in Appendix A.

Lemma 3.2 (Absorption lemma). *Let $2 \leq k < t$ be integers and let $\gamma > 0$. Then, there is an integer n_0 satisfying the following: for each balanced t -partite k -graph H with each class of size $n \geq n_0$ and*

$$\tilde{\delta}_{k-1}(H) \geq \left(1 - \left(\binom{t-2}{k-1} + 2 \binom{t-2}{k-2} \right)^{-1} + \gamma \right) n,$$

there exists a balanced vertex subset $U \subset V(H)$ of size $|U| \leq \gamma^{t(t-1)} n / 2^{t+1}$ such that there exists a perfect K_t^k -matching in $H[U \cup W]$ for every balanced vertex subset $W \subset V \setminus U$ of size $|W| \leq \gamma^{2t(t-1)} n / 2^{2t+4}$.

Our next task is to find a large K_t^k -matching in H covering all but at most ϵn vertices, which requires a theorem of Frankl and Rödl [7] and Chernoff's inequality. The proof of Lemma 3.5 is based on Claim 4.1 in [2].

Theorem 3.3 (Frankl and Rödl [7]). *For all $k, \epsilon \geq 0$ and $a > 3m$ there exists $\tau = \tau(\epsilon)$, $D = D(n)$, and $n_0 = n_0(\tau)$ such that if $n \geq n_0$ and H is a k -graph of order n satisfying*

1. $\deg^H(v) = (1 \pm \tau)D$ for all $v \in V$, and
2. $\Delta_2(H) = \max_{T \in \binom{V(H)}{2}} \deg^H(T) < D/(\log n)^a$

then H contains a matching M covering all but at most ϵn vertices.

Lemma 3.4 (Chernoff's inequality (see e.g. [3])). *Let $X \sim \text{Bin}(n, p)$. Then*

$$\mathbb{P}(|X - np| \geq \lambda) \leq 2 \exp\left(-\frac{\lambda^2}{2np}\right) \text{ and } \mathbb{P}(X - np \leq \lambda) \leq \exp\left(-\frac{\lambda^2}{2np}\right).$$

Lemma 3.5. *Let $2 \leq k \leq t$ be integers with $\epsilon > 0$. Then, there exist an integer n_0 and a constant $\gamma > 0$ such that every t -partite k -graph H with partition classes V_1, \dots, V_t , each of size $n > n_0$ and*

$$\tilde{\delta}_{k-1}(H) \geq \phi^{*,k}(t, n) + \gamma n$$

contains a K_t^k -matching \mathcal{T} covering all but at most ϵn vertices.

Proof. Fix k, t and ϵ . Write $\phi^* = \phi^{*,k}(t, n)/n$. We assume that n is sufficiently large throughout the proof. Let H be a balanced t -partite k -graph H with partition classes V_1, \dots, V_t , each of size n and $\tilde{\delta}_{k-1}(H) \geq (\phi^* + \gamma)n$. Our aim is to define a t -graph H^* on vertex set $V(H)$ satisfying the condition of Theorem 3.3, where every edge in H^* corresponds to a K_t^k in H . Hence, by Theorem 3.3, there exists a matching M covering all but at most ϵn vertices of H^* corresponding to a K_t^k -matching in H .

We are going to construct H^* via two rounds of randomisation. Let R_i be a random binomial subset of V_i with probability $p = n^{-0.9}$ for $i \in [t]$. Let $R = (R_1, \dots, R_t)$. Hence, by Chernoff's inequality (Lemma 3.4)

$$\mathbb{P}(|R_i - n^{0.1}| \geq n^{0.075}) \leq 2 \exp(-n^{0.05}/2). \quad (5)$$

For each $I \in \binom{[t]}{k-1}$, each I -legal set $T \subset R$ and $i \in [t] \setminus I$

$$\mathbb{E}(\deg_i^{H[I]}(T)) \geq (\phi^* + \gamma)n \times n^{-0.9} = (\phi^* + \gamma)n^{0.1}.$$

Again, by Chernoff's inequality (Lemma 3.4)

$$\mathbb{P}(\deg_i^{H[I]}(T) < (\phi^* + \gamma/2)n^{0.1}) \leq \exp(-\gamma^2 n^{0.1}/(8(\phi^* + \gamma))). \quad (6)$$

Let $m = n^{0.1} - n^{0.075}$. Let R'_i be a randomly chosen m -set in R_i and let $R' = (R'_1, \dots, R'_t)$. By (5) and (6), we have with probability $1 - e^{-\Omega(n^{0.05})}$

$$\tilde{\delta}_{k-1}(H[R']) \geq (c + \gamma/2)n^{0.1} - 2n^{0.075} \geq (c + \gamma/4)m.$$

In addition,

$$n^{-0.9} \geq \mathbb{P}(v \in R') \geq n^{-0.9} \frac{\binom{m+2n^{0.075}-1}{m-1}}{\binom{m+2n^{0.075}}{m}} \geq (1 - 2n^{-0.025})n^{-0.9}.$$

Now, we take $n^{1.1}$ independent copies of R' and denote them by $R^1, \dots, R'^{n^{1.1}}$. For a subset of vertices $S \subset V$, let

$$Y_S = |\{i : S \subset R'^i\}|.$$

Note that $\mathbb{E}(Y_S) \leq n^{1.1-0.9|S|}$. With probability at least $1 - 2\exp(-9n^{1.5}/2)$ by Lemma 3.4, $Y_v = n^{0.2} \pm 3n^{0.175}$ for every $v \in V$, where $y = x \pm c$ means that $x - c \leq y \leq x + c$. Let $Z_2 = |\{S \in \binom{V}{2} : Y_S \geq 3\}|$ and observe that

$$\mathbb{E}(Z_2) < n^2 (n^{1.1})^3 (n^{-0.9})^6 = n^{-0.1}.$$

For $k \geq 3$, let $Z_k = |\{S \in \binom{V}{k} : Y_S \geq 2\}|$ and observe that

$$\mathbb{E}(Z_k) < n^k (n^{1.1})^2 (n^{-0.9})^{2k} = n^{-0.2}.$$

The latter implies that every 3-set $S \in \binom{V}{3}$ lies in at most one R'^i with high probability. In summary, there exist $n^{1.1}$ vertex sets $R'^1, \dots, R'^{n^{1.1}}$ such that

- (i) for every $v \in V$, $Y_v = n^{0.2} \pm 3n^{0.175}$,
- (ii) every 2-set $S \in \binom{V}{2}$ is in at most two sets R'^i ,
- (iii) every 3-set $S \in \binom{V}{3}$ is in at most one set R'^i ,
- (iv) for $i \in [n^{1.1}]$, $R'^i = (R'_1{}^i, \dots, R'_q{}^i)$ with $R'_j{}^i \subset V_j$ and $|R'_j{}^i| = m$ for $j \in [q]$,
- (v) for $i \in [n^{1.1}]$, $\tilde{\delta}_{k-1}(H[R'^i]) \geq (\phi^* + \gamma/4)m$.

Fix one such sequence $R'^1, \dots, R'^{n^{1.1}}$.

By (v) and Theorem 1.6, there exists a fractional perfect K_t^k -matching w^i in $H[R'^i]$ for $i \in [n^{1.1}]$. Now we conduct our second round of random process by defining a random t -graph H^* on vertex classes V such that each $[t]$ -legal set T is randomly independently chosen with

$$\mathbb{P}(T \in H^*) = \begin{cases} w^{i_T}(T) & \text{if } T \in \mathcal{K}_t^k(H[R'^{i_T}]) \text{ for some } i_T \in [t], \\ 0 & \text{otherwise.} \end{cases}$$

Note that i_T is unique by (iii) and so H^* is well defined. For $v \in V$, let $I_v = \{i : v \in R'^i\}$ and so $|I_v| = Y_v = n^{0.2} \pm 3n^{0.175}$ by (i). For every $v \in V$, let E_v^i be the set of K_t^k in $H[R'^i]$ containing v . Thus, for $v \in V$, $\deg^{H^*}(v)$ is a generalised binomial random variable with expectation

$$\mathbb{E}(\deg^{H^*}(v)) = \sum_{i \in I_v} \sum_{T \in E_v^i} w^i(T) = |I_v| = n^{0.2} \pm 3n^{0.175}.$$

Similarly, for every 2-set $\{u, v\}$,

$$\mathbb{E}(\deg^{H^*}(u, v)) = \sum_{i \in I_v \cap I_u} \sum_{T \in E_v^i \cap E_u^i} w^i(T) \leq |I_v \cap I_u| = |Y_{\{u, v\}}| \leq 2,$$

by (ii). Hence, again by Chernoff's inequality, Lemma 3.4, we may assume that for every $v \in V$ and every 2-set $\{u, v\}$

$$\deg^{H^*}(v) = n^{0.2} \pm 4n^{0.2-\epsilon}, \deg^{H^*}(u, v) < n^{0.1}.$$

Thus, H^* satisfies the hypothesis of Theorem 3.3 and the proof is completed. \square

Next we prove Theorem 1.3 and Theorem 1.5.

Proof of Theorem 1.3 and Theorem 1.5. Fix k and t and $\gamma > 0$. Let

$$d = \begin{cases} (t-1)/t & \text{if } k = 2 \\ 1 - \left(\binom{t-1}{k-1} + 2\binom{t-2}{k-2} \right)^{-1} & \text{if } k \geq 3. \end{cases}$$

Note that $d \geq \phi^{*,k}(t, n)$ by Theorem 1.6. Let H be a t -partite k -graph with vertex classes V_1, \dots, V_t each of size $n \geq n_0$ and $\tilde{\delta}_{k-1}(H) \geq (d + \gamma)n$.

We are going to show that H contains a perfect K_t^k -matching. Throughout this proof, n_0 is assumed to be sufficiently large. There exists a balanced vertex set U in V satisfying the conditions of the absorption lemma, Lemma 3.2. Set $H' = H[V \setminus U]$ and note that $\tilde{\delta}_{k-1}(H') \geq (d + \gamma/2)n \geq (\phi^{*,k}(t, n) + \gamma/2)n$. There exists a K_t^k -matching \mathcal{T} in H' covering all but at most ϵn vertices of H' for some small $\epsilon > 0$. Let $W = V(H') \setminus V(\mathcal{T})$, so W is balanced. Since $H[U \cup W]$ contains a perfect K_t^k -matching \mathcal{T}' by the choice of U , $\mathcal{T} \cup \mathcal{T}'$ is a perfect K_t^k -matching in H . \square

4 Proof of Theorem 1.7

Note that together Lemma 3.5 and the lemma below imply Theorem 1.7. Hence all that remains is to prove Lemma 4.1.

Lemma 4.1. *For integers $t \geq k \geq 2$, there exists n_0 such that the following holds. Suppose that H is a k -graph with $n \geq n_0$ vertices with $t|n$. Then there exists a partition V_1, \dots, V_t of $V(H)$ into sets of size n/t such that for every $l \in [k-1]$, every $I \in \binom{[t]}{l}$, every legal I -set T and $J \in \binom{[t] \setminus I}{k-l}$, we have*

$$\frac{t^{k-l}}{(k-l)!} \deg_J^{H'}(T) \geq \deg^H(T) - 2(t \ln n)^{1/2} n^{k-l-1/2},$$

where H' is the induced t -partite k -subgraph of H with vertex classes V_1, \dots, V_t .

Proof. First set $m = k - l$ and let U_1, \dots, U_t be a random partition of V , where each vertex appears in vertex class U_j independently with probability $1/t$. For a fixed l -set $T = \{v_1, \dots, v_l\}$, we identify $N^H(T)$ to be the link hypergraph of T . Thus, $N^H(T)$ is an m -graph with $\deg^H(T)$ edges. We decompose the $N^H(T)$ into at most $i_0 \leq mn^{m-1}$ nonempty pairwise edge disjoint matching, which denote by M_1, \dots, M_{i_0} .

For every edge $E \in N^H(T)$, and every index set $J \in \binom{[t]}{m}$, we say that E is J -good, if E is J -legal with respect to U_1, \dots, U_t . Since the partition U_1, \dots, U_t was chosen randomly, we have for fixed $J \in \binom{[t]}{m}$

$$\mathbb{P}(E \text{ is good}) = m!t^{-m}.$$

Thus, for $X_{i,J} = X_{i,J}(T) = |\{E \in M_i : E \text{ is } J\text{-good}\}|$ we have

$$\mu_{i,J} = \mu_{i,J}(T) = \mathbb{E}(X_{i,J}) = \frac{m!}{t^m} |M_i|.$$

Now call a matching M_i *bad* (with respect to U_1, \dots, U_t) if there exists a set $J \in \binom{[t]}{m}$ such that

$$X_{i,J} \leq \left(1 - \left(\frac{2(2k-1) \ln n}{\mu_{i,J}}\right)^{1/2}\right) \mu_{i,J}$$

and call T a *bad set* if there is at least one bad $M_i = M_i(T)$. Otherwise call T a *good set*. For a fixed M_i the event ' E is J -good' with $E \in M_i$ are jointly independent, hence by Chernoff's inequality, Lemma 3.4,

$$\mathbb{P}(M_i \text{ is bad}) \leq \binom{t}{m} \exp(-(2k-1) \ln n) = \binom{t}{m} n^{-2k+1}.$$

Recall that $i_0 \leq mn^{m-1}$ and $m \leq k-1$, we have

$$\mathbb{P}(T \text{ is bad}) \leq i_0 \binom{t}{m} n^{-2k+1} \leq n^{-k}$$

and by summing over all l -sets T we obtain that

$$\mathbb{P}(\text{there exists a bad } l\text{-set}) \leq n^{-1}.$$

Moreover, Chernoff's inequality, Lemma 3.4, yields

$$\mathbb{P}(|U_j| \geq n/t + n^{1/2}(\ln n)^{1/4}/t) \leq \exp(-(\ln n)^{1/2}/t).$$

Thus with positive probability there is a partition U_1, \dots, U_t such that all l -sets T are good and

$$|U_j| \leq n/t + n^{1/2}(\ln n)^{1/4}/t \text{ for all } j \in [t].$$

Consequently, by redistributing at most $n^{1/2}(\ln n)^{1/4}/t$ vertices of the partition U_1, \dots, U_t we obtain an equipartition V_1, \dots, V_t with

$$|V_j| = n/t \text{ and } |U_j \setminus V_j| \leq n^{1/2}(\ln n)^{1/4}/t \text{ for all } j \in [t].$$

Let H' be the induced t -partite k -subgraph with vertex classes V_1, \dots, V_t . Note that for an l -set $I \in \binom{[t]}{l}$, a I -legal set T and an m -set $J \in \binom{[t] \setminus I}{m}$,

$$\begin{aligned} \deg_J^{H'}(T) &\geq \sum_{i \in [i_0]} \left(1 - \left(\frac{2(2k-1) \ln n}{\mu_{i,J}} \right) \right) \mu_{i,J} - m \frac{n^{1/2}(\ln n)^{1/4}}{t} n^{m-1} \\ &\geq \frac{m!}{t^m} \deg_J^H(T) - (2(2k-1) \ln n)^{1/2} \sum_{i \in [i_0]} \mu_{i,J}^{1/2} - m \frac{n^{1/2}(\ln n)^{1/4}}{t} n^{m-1}. \end{aligned}$$

By the Cauchy-Schwarz inequality, we obtain that

$$\sum_{i \in [i_0]} \mu_{i,J}^{1/2} \leq \left(i_0 \sum_{i \in [i_0]} \mu_{i,J} \right)^{1/2} \leq \left(mn^{m-1} \frac{m!}{t^m} \binom{n}{m} \right)^{1/2} \leq n^{m-1/2}$$

Therefore,

$$\deg_J^H(T) \geq \frac{m!}{t^m} \deg_J^H(T) - 2(k \ln n)^{1/2} n^{m-1/2},$$

as required. \square

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A Proof of Lemma 3.2

Proof. Throughout the proof we may assume that n_0 is chosen sufficiently large. Let H be a balanced t -partite k -graph with partition classes V_1, \dots, V_k each of size n and minimum l -degree $\delta_l(H) \geq (1/2 + \gamma)n^{k-l}$. Define H' be a t -partite t -graph on V_1, \dots, V_k and $v_1 v_2 \dots v_t \in E(H')$ if and only if $v_1 v_2 \dots v_t$ is a K_t^k in H . Furthermore set $m = t(t-1)$ and call a balanced m -set A an *absorbing* m -set for a balanced t -set T if A spans a matching of size $t-1$ in H' and $A \cup T$ spans a matching of size t in

H' , in other words, $A \cap T = \emptyset$ and both $H'[A]$ and $H'[A \cup T]$ contain a perfect matching. Denote by $\mathcal{L}(T)$ the set of all absorbing m -sets for T . Next, we show that for every balanced k -set T , there are many absorbing m -sets for T .

Claim A.1. *For every balanced k -set T , $|\mathcal{L}(T)| \geq (\gamma)^m \binom{n}{t-1}^t / 2^t$.*

Proof. Let $T = \{v_1, \dots, v_t\}$ be fixed with $v_i \in V_i$ for $i \in [t]$. By Proposition 3.1, it is easy to see that there exist at least $(\gamma n)^{t-1}$ edges in H' containing v_1 . Since n_0 was chosen large enough, there are at most $(t-1)n^{t-2} \leq (\gamma n)^{t-1}/2$ edges in H' , which contain v_1 and v_j for some $2 \leq j \leq t$. We fix one such edge $\{v_1, u_2, \dots, u_t\}$ with $u_j \in V_j \setminus \{v_j\}$ for $2 \leq j \leq t$. Set $U_1 = \{u_2, \dots, u_t\}$ and $W_0 = T$. For each $2 \leq j \leq t$ and each pair u_j, v_j suppose we succeed to choose a $(k-1)$ -set U_j such that U_j is disjoint to $W_{j-1} = U_{j-1} \cup W_{j-2}$ and both $U_j \cup \{u_j\}$ and $U_j \cup \{v_j\}$ are edges in H' . Then for a fixed $2 \leq j \leq t$ we call such a choice U_j *good*, motivated by $A = \bigcup_{j \in [t]} U_j$ being an absorbing m -set for T .

Note that in each step $2 \leq j \leq t$ there are $t + (j-1)(t-1)$ vertices in W_{j-1} . More specifically, there are at most $j \leq k$ vertices in $V_i \cap W_{j-1}$ for $i \in [t]$. Thus, the number of edges in H' intersecting u_j (or v_j respectively) and at least one other vertex in W_{j-1} is at most $(t-1)jn^{t-2} < t^2 n^{t-2} \leq (\gamma n)^{t-1}/2$. For each $2 \leq j \leq t$, by Proposition 3.1 there are at least $(\gamma n)^{t-1} - (\gamma n)^{t-1}/2 = (\gamma n)^{t-1}/2$ choices for U_j and in total we obtain $(\gamma n)^m / 2^t$ absorbing m -sets for T with multiplicity at most $((t-1)!)^t$. \square

Now, choose a family F of balanced m -sets by selecting each of the $\binom{n}{t-1}^t$ possible balanced m -sets independently with probability

$$p = \gamma^m n / \left(t 2^{t+2} \binom{n}{t-1}^t \right).$$

Then, by Chernoff's inequality, Lemma 3.4 with probability $1 - o(1)$ as $n \rightarrow \infty$, the family F satisfies the following properties:

$$|F| \leq \gamma^m n / (t 2^{t+1}) \quad (7)$$

and

$$|\mathcal{L}(T) \cap F| \geq \frac{\gamma^{2m} n}{t 2^{2t+3}} \quad (8)$$

for all balanced t -sets T . Furthermore, we can bound the expected number of intersecting m -sets by

$$\binom{n}{t-1}^t \times t(t-1) \times \binom{n}{t-2} \binom{n}{t-1}^{t-1} \times t^2 \leq \frac{\gamma^{2m} n}{t 2^{2t+4}}$$

Thus, using Markov's inequality, we derive that with probability at least $1/2$

$$F \text{ contains at most } \gamma' n \text{ intersecting pairs.} \quad (9)$$

Hence, with positive probability the family F has all properties stated in (7), (8) and (9). By deleting all the intersecting balanced m -sets and

non-absorbing m -sets in such a family F , we get a subfamily F' consisting of pairwise disjoint balanced m -sets, which satisfies

$$|\mathcal{L}(T) \cap F'| \geq \frac{\gamma^{2m}n}{t2^{2t+3}} - \frac{\gamma^{2m}n}{t2^{2t+4}} \geq \frac{\gamma^{2m}n}{t2^{2t+4}}$$

for all balanced t -sets T . Let $U = V(F')$ and so U is balanced of size $|U| \leq \gamma^m n / (2^{t+1})$. For a balanced set $W \subset V \setminus V(M)$ of size $|W| \leq \frac{\gamma^{2m}n}{2^{2t+4}}$, W can be partitioned into at most $\frac{\gamma^{2m}n}{t2^{2t+4}}$ balanced t -sets. Each balanced t -set can be successively absorbed using a different absorbing m -set in F' , so there exists a perfect matching in $H'[U \cup W]$. Hence, there is a perfect K_t^k -matching in $H[U \cup W]$. \square